

Magic identities for conformal four-point integrals

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ABSTRACT: We propose an iterative procedure for constructing classes of off-shell four-point conformal integrals which are identical. The proof of the identity is based on the conformal properties of a subintegral common for the whole class. The simplest examples are the so-called ‘triple scalar box’ and ‘tennis court’ integrals. In this case we also give an independent proof using the method of Mellin–Barnes representation which can be applied in a similar way for general off-shell Feynman integrals.

KEYWORDS: Supersymmetric gauge theory, NLO Computations, Conformal and W Symmetry.

*UMR 5108 associée à l'Université de Savoie

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1. Introduction

Four-point correlators in the $\mathcal{N} = 4$ super-Yang-Mills conformal field theory have attracted considerable attention since the formulation of the AdS/CFT conjecture [1]. They can provide non-trivial dynamical information about the CFT side of the correspondence, which can then be compared to its AdS dual. In particular, the correlators of four ‘protected’ stress-tensor multiplets have been extensively studied. It has been found that their form is more restricted than would follow from just superconformal kinematics. This property, called ‘partial non-renormalisation’ in [2] is observed in the perturbative one-loop [3] and two-loop [4] CFT calculations, as well as in their AdS supergravity (or strong coupling) dual [5]. These explicit results have been analysed through OPE methods [6] and the two-loop anomalous dimensions of all twist two operators in the theory were found in [7]. In all these studies conformal four-point integrals have been instrumental.

In a parallel development, on-shell four-gluon planar scattering amplitudes in $\mathcal{N} = 4$ SYM have been investigated in [8] and a remarkable conjecture about their iterative structure has been made, based on the comparison of one- and two-loop results. The conjecture was confirmed at three loops in [9]. If true to all orders, this iterative property may allow the resummation of the perturbative series and may be the manifestation of some form of integrability of the theory. One of the results of [9] was the large spin asymptotic value of the anomalous dimension of twist two operators, in agreement with the conjectured three-loop formula of [10]. The latter also received impressive confirmation from the integrable model proposed in [11].

Although it may seem that the two problems, that of the correlators of gauge-invariant composites and that of gluon scattering amplitudes, are unrelated, it is quite significant that in both studies one deals with the same conformal four-point integrals. Up to two loops, these are the so-called ‘scalar box’ (or ‘ladder’) integrals.¹ At three loops, in addition to the triple scalar box a new integral named ‘tennis court’ has appeared in [9]. In the context

¹The off-shell ladder integrals in four dimensions for an arbitrary number of loops have been evaluated in [13, 14] and generalised to arbitrary dimensions in [15].

of the scattering amplitudes these two integrals are put on the massless shell, whereby they become infrared and collinear divergent. Their pole structure in dimensional regularisation is quite different, as shown in [9]. In the present paper we prove that the two integrals, considered *off shell*, are identical. We first show this by a very simple argument, based on a ‘turning symmetry’ property of the two-loop scalar box subintegral common for both three-loop integrals. It should be stressed that our proof requires conformal invariance in strictly four dimensions, therefore it does not apply to the dimensionally regularised on-shell version of the integrals. To rule out the possibility of contact terms spoiling the proof we give an alternative argument which relates the two three-loop integrals to the same four-loop integral under the action of a differential operator. We then present a simple graphical rule for constructing identical integrals which is easy to iterate to any number of loops. In some sense our iteration procedure (or ‘slingshot rule’) resembles the so-called ‘rung rule’ of [8, 9]. Thus, at four loops we produce five apparently different, but in fact identical integrals obtained by iterating the already established three-loop identity of the scalar box and the tennis court. We then give an independent confirmation of the latter by explicitly computing the two integrals using the Mellin–Barnes method.

2. Conformal four-point integrals

We will discuss an infinite class of conformal four-point integrals in four dimensions², each of which is essentially described by a function of two variables. We begin with the simplest example, the one-loop ladder integral,

$$h^{(1)}(x_1, x_2, x_3, x_4) = \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \frac{1}{x_{13}^2 x_{24}^2} \Phi^{(1)}(s, t). \quad (2.1)$$

Here $x_{ij} = x_i - x_j$ and the conformal cross-ratios s and t are

$$s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad t = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (2.2)$$

The fact that the integral is characterised by a single function of two variables follows from its conformal covariance [16]. Indeed, performing a conformal inversion on all points,

$$x^\mu \longrightarrow \frac{x^\mu}{x^2} \implies x_{ij}^2 \longrightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}, \quad d^4 x_5 \longrightarrow \frac{d^4 x_5}{x_5^8}, \quad (2.3)$$

we find that the integral transforms covariantly with weight one at each point,

$$h^{(1)}(x_1, x_2, x_3, x_4) \longrightarrow x_1^2 x_2^2 x_3^2 x_4^2 h^{(1)}(x_1, x_2, x_3, x_4). \quad (2.4)$$

Since rotation and translation invariance are manifest, we conclude that the integral is given by a conformally covariant combination of propagators multiplied by a function of the conformally invariant cross-ratios (2.2).

²In this section we consider and prove identities for Euclidean integrals. The corresponding Minkowskian version of the identities can be obtained through Wick rotation of the integrals. In the Euclidean context we consider integrals with separated external points, $x_{ij} \neq 0$. This is the Euclidean analogue of the off-shell regime, $x_{ij}^2 \neq 0$, for a Minkowskian integral.

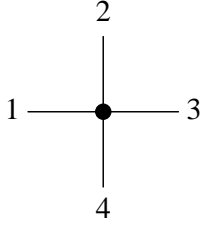


Figure 1: The one-loop ladder integral. Each line represents a propagator with the integration point given by a solid vertex. The reason for the names ladder and box is clearer in the momentum representation of the same integral.

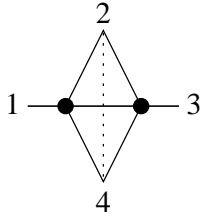


Figure 2: The two-loop ladder integral. The dashed line represents the numerator x_{24}^2 .

The function $\Phi^{(1)}(s, t)$ has been calculated in [12, 13], where it was also shown that the same function appears in a three-point integral. The latter can be obtained from the four-point one by sending one of the points to infinity [16]. We can multiply equation (2.1) by x_{13}^2 , say, and then take the limit $x_3 \rightarrow \infty$. This gives,

$$h_{3\text{pt}}^{(1)}(x_1, x_2, x_4) = \lim_{x_3 \rightarrow \infty} x_{13}^2 h^{(1)}(x_1, x_2, x_3, x_4) = \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{45}^2} = \frac{1}{x_{24}^2} \Phi^{(1)}(\hat{s}, \hat{t}), \quad (2.5)$$

where the cross-ratios s and t have become \hat{s} and \hat{t} in the limit,

$$s \longrightarrow \hat{s} = \frac{x_{12}^2}{x_{24}^2}, \quad t \longrightarrow \hat{t} = \frac{x_{14}^2}{x_{24}^2}. \quad (2.6)$$

Thus the three-point integral contains the same information as the four-point integral, i.e. the same function of two variables. The reason is that one can use translations and conformal inversion to take the point x_3 to infinity and the function of the cross-ratios is invariant under these transformations.

The integral (2.1) is the first in an infinite series of conformal integrals, the n -loop ladder (or scalar box) integrals, which have all been evaluated [14]. In particular the 2-loop ladder integral is given by

$$h^{(2)}(x_1, x_2, x_3, x_4) = x_{24}^2 \int \frac{d^4 x_5 d^4 x_6}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{26}^2 x_{46}^2 x_{36}^2} = \frac{1}{x_{13}^2 x_{24}^2} \Phi^{(2)}(s, t). \quad (2.7)$$

The prefactor x_{24}^2 is present to give conformal weight one at each external point.

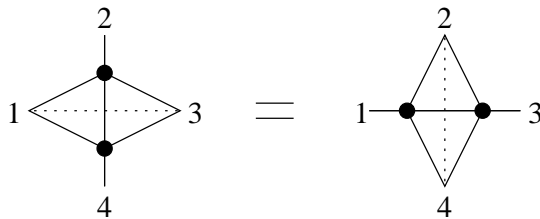


Figure 3: The two-loop turning identity obtained from the pairwise point swap, $x_1 \longleftrightarrow x_2$, $x_3 \longleftrightarrow x_4$.

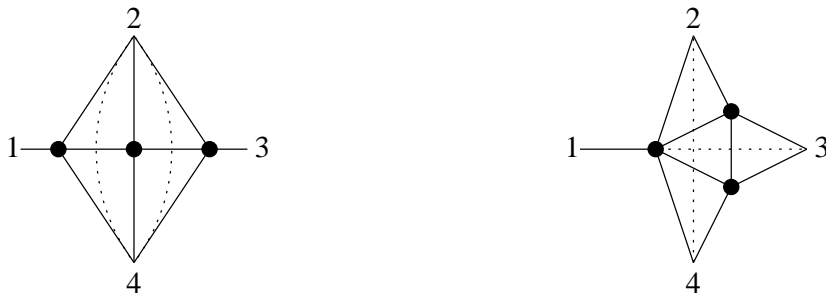


Figure 4: Two examples of three-loop conformal four-point integrals, the three-loop ladder and the ‘tennis-court’.

Again conformal transformations can be used to justify the appearance of the 2-variable function $\Phi^{(2)}$. The r.h.s. of (2.7) is invariant under the pairwise swap $x_1 \longleftrightarrow x_2$, $x_3 \longleftrightarrow x_4$, hence

$$h^{(2)}(x_2, x_1, x_4, x_3) = h^{(2)}(x_1, x_2, x_3, x_4). \quad (2.8)$$

This symmetry is not immediately evident from the integral. It is its conformal nature which allows this identification.

At three loops we consider two conformal integrals, the three-loop ladder,

$$h^{(3)}(x_1, x_2, x_3, x_4) = x_{24}^4 \int \frac{d^4 x_5 d^4 x_6 d^4 x_7}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{26}^2 x_{46}^2 x_{67}^2 x_{27}^2 x_{47}^2 x_{37}^2} = \frac{1}{x_{13}^2 x_{24}^2} \Phi^{(3)}(s, t), \quad (2.9)$$

and the so-called ‘tennis court’ [9],

$$g^{(3)}(x_1, x_2, x_3, x_4) = x_{24}^2 \int \frac{x_{35}^2 d^4 x_5 d^4 x_6 d^4 x_7}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{57}^2 x_{67}^2 x_{26}^2 x_{47}^2 x_{36}^2 x_{37}^2} = \frac{1}{x_{13}^2 x_{24}^2} \Psi^{(3)}(s, t) \quad (2.10)$$

Notice the presence of the numerator x_{35}^2 in the integrand of the tennis court. It is needed to balance the conformal weight of the five propagators coming out of point 5.

We will show that the three-loop ladder and the tennis court are in fact the same, i.e. we will prove $\Phi^{(3)} = \Psi^{(3)}$. First we shall present a diagrammatic argument. We consider the n -loop ladder as being iteratively constructed from the $(n - 1)$ -loop ladder by integrating against a ‘slingshot’ (the ‘0-loop’ ladder is a product of free propagators). For

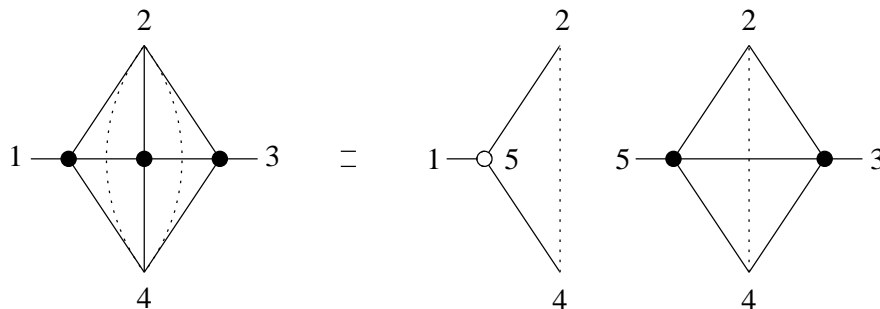


Figure 5: The three-loop ladder expressed as the integral of the two-loop ladder against the ‘slingshot’. The empty vertex is the point x_5 which must be identified with the point x_5 from the two-loop ladder sub-integral before being integrated over.

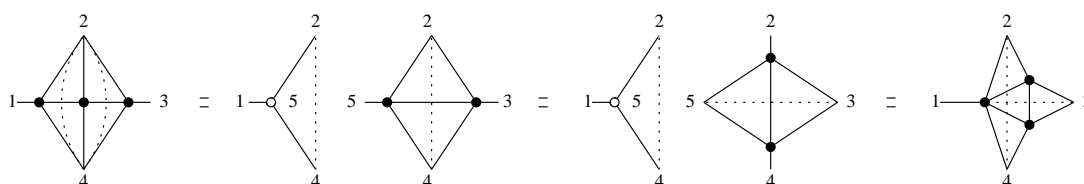


Figure 6: Diagrammatic representation of the proof of equality of the tennis court and the three-loop ladder. The identity follows from the turning identity (2.8) for the two-loop subintegral.

example we write the three-loop ladder as

$$h^{(3)}(x_1, x_2, x_3, x_4) = x_{24}^2 \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{45}^2} \left(x_{24}^2 \int \frac{d^4 x_6 d^4 x_7}{x_{56}^2 x_{26}^2 x_{46}^2 x_{67}^2 x_{27}^2 x_{47}^2 x_{37}^2} \right), \quad (2.11)$$

where inside the parentheses we recognise the two-loop ladder integral (2.7).

We can then show the equality of the three-loop ladder and the tennis court by using the turning symmetry (2.8) on the two-loop ladder sub-integral. Then the tennis court integral (2.10) can be recognised as the turned two-loop ladder integrated against the slingshot,

$$\begin{aligned} h^{(3)}(x_1, x_2, x_3, x_4) &= x_{24}^2 \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{45}^2} h^{(2)}(x_5, x_2, x_3, x_4), \\ &= x_{24}^2 \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{45}^2} h^{(2)}(x_2, x_5, x_4, x_3), \\ &= g^{(3)}(x_1, x_2, x_3, x_4). \end{aligned} \quad (2.12)$$

This proof can be more easily seen in the diagram (Fig. 6).

In using the turning identity (2.8) we have ignored the possibility of contact terms. These could, in principle, spoil the derivation of identities like $\Phi^{(3)} = \Psi^{(3)}$ as the proof (2.12) involves turning a subintegral. Contact terms could then generate regular terms upon doing one further integration. We now give an argument why this cannot happen for any conformal four-point integral. We again use the example of the 3-loop ladder and tennis court identity.

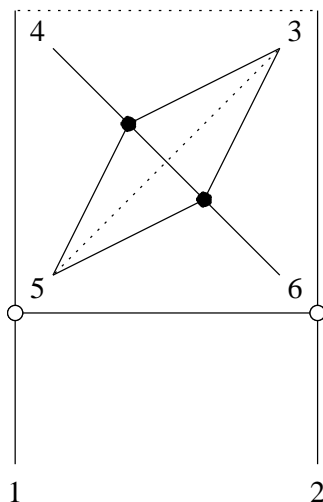


Figure 7: The 2-loop ladder inserted into an H-shaped frame, generating a 4-loop integral.

Consider inserting the n -loop subintegral (the 2-loop ladder in this case) into an H-shaped frame with a dashed line across the top, as illustrated below. This generates an $(n + 2)$ -loop integral which is conformal with weight 1 at each external point (provided the subintegral is conformal with weight 1 at each external point).

When inserting the 2-loop ladder in this way the 4-loop integral one obtains is

$$\begin{aligned}
 f^{(4)}(x_1, x_2, x_3, x_4) &= x_{34}^2 \int \frac{d^4 x_5 d^4 x_6}{x_{15}^2 x_{45}^2 x_{56}^2 x_{26}^2 x_{36}^2} x_{35}^2 \int \frac{d^4 x_7 d^4 x_8}{x_{67}^2 x_{57}^2 x_{37}^2 x_{78}^2 x_{58}^2 x_{38}^2 x_{48}^2} \\
 &= \frac{1}{x_{13}^2 x_{24}^2} f(s, t).
 \end{aligned}
 \tag{2.13}$$

As usual, the second equality follows from conformality.

Now we consider the action of \square_1 on the above integral using

$$\square \frac{1}{x^2} = -4\pi^2 \delta(x).
 \tag{2.14}$$

On the integral one obtains

$$-4\pi^2 \frac{x_{34}^2 x_{13}^2}{x_{14}^2} \int \frac{d^4 x_6 d^4 x_7 d^4 x_8}{x_{26}^2 x_{16}^2 x_{36}^2 x_{67}^2 x_{17}^2 x_{37}^2 x_{78}^2 x_{18}^2 x_{38}^2 x_{48}^2} = -\frac{4\pi^2 x_{34}^2}{x_{13}^4 x_{14}^2 x_{24}^2} \Phi^{(3)}(s, t).
 \tag{2.15}$$

On the functional form of (2.13) one uses the chain rule to derive the action of a differential operator on the function f . In this way we find the differential equation,

$$\frac{x_{23}^2 x_{34}^2}{x_{13}^6 x_{24}^4} \Delta_{st}^{(2)} f(s, t) = -\frac{\pi^2 x_{34}^2}{x_{13}^4 x_{14}^2 x_{24}^2} \Phi^{(3)}(s, t).
 \tag{2.16}$$

The operator $\Delta_{st}^{(2)}$ is given explicitly by

$$\Delta_{st}^{(2)} = s\partial_s^2 + t\partial_t^2 + (s+t-1)\partial_s\partial_t + 2\partial_s + 2\partial_t. \tag{2.17}$$

Similarly we can act with \square_2 on the 4-loop integral to obtain the following integral,

$$-4\pi^2 \frac{x_{34}^2}{x_{23}^2} \int \frac{d^4x_5 d^4x_7 d^4x_8 x_{35}^2}{x_{15}^2 x_{25}^2 x_{45}^2 x_{57}^2 x_{58}^2 x_{78}^2 x_{27}^2 x_{48}^2 x_{37}^2 x_{38}^2} = -\frac{4\pi^2 x_{34}^2}{x_{23}^2 x_{13}^2 x_{24}^4} \Psi^{(3)}(s, t), \tag{2.18}$$

and the corresponding differential equation,

$$\frac{x_{14}^2 x_{34}^2}{x_{24}^6 x_{13}^4} \Delta_{st}^{(2)} f(s, t) = -\frac{\pi^2 x_{34}^2}{x_{23}^2 x_{13}^2 x_{24}^4} \Psi^{(3)}(s, t). \tag{2.19}$$

From (2.16,2.19) it follows that $\Phi^{(3)} = \Psi^{(3)}$, the point being that one obtains the *same* differential operator $\Delta_{st}^{(2)}$ under the two \square operations. The argument has the obvious generalisation of placing any conformal integral (in any orientation) inside the frame. This argument indirectly shows that the previous argument (2.12) based on turning the subintegral cannot suffer from contact term contributions.

The identity we have obtained at three loops is just the first example of an infinite set of identities which all come from the turning symmetry of subintegrals. We generate $(n+1)$ -loop integrals by integrating n -loop integrals against the slingshot in all possible orientations. The resulting integrals are equal by turning identities of the form (2.8). At two loops we get just one integral (the two-loop ladder). At three loops we have already seen two equivalent integrals (ladder and tennis court). At four loops we generate two equivalent integrals from the three-loop ladder and three equivalent integrals from the tennis court. Finally, all five four-loop integrals obtained in this way are equivalent by the three-loop identity for the ladder and tennis court (see Fig. 8).

In general it is more common to give the diagrams in the ‘momentum’ representation (which has nothing to do with the Fourier transform) where we regard the integrations as integrals over loop momenta rather than coordinate space vertices. This representation is neater but the numerators need to be described separately as they do not appear in the diagrams. To return to the coordinate space integrals one places a vertex inside each loop and connects them with propagators through each line. We show this in Fig. 9 for the tennis court integral. The momentum-space version of the four generations of integrals from Fig. 8 is then given in Fig. 10.

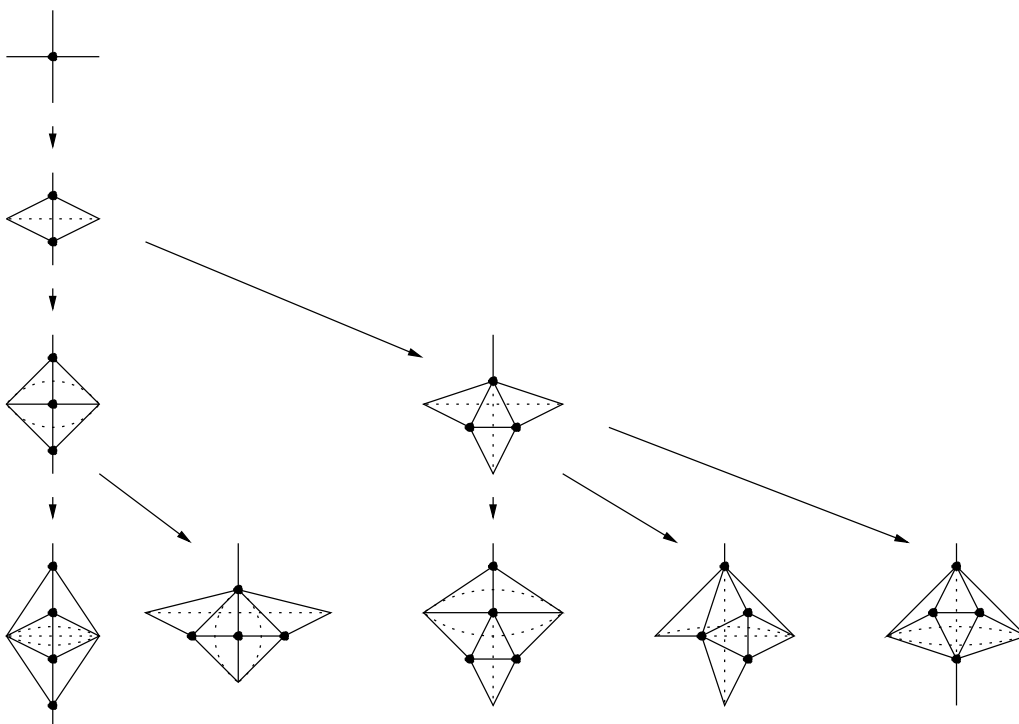


Figure 8: The integrals in a given row are all equivalent. They generate the integrals in the next row by being integrated in all possible orientations against the slingshot attached from above. The ladder series is in the left-most column.

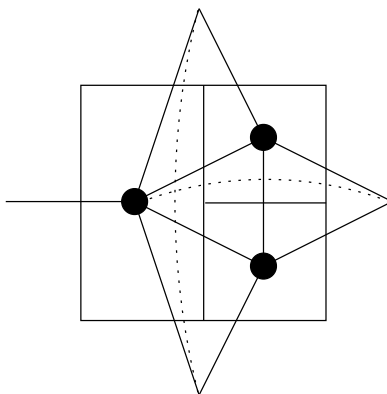


Figure 9: The conversion from the momentum notation to the coordinate space notation. The pictures represent the same integral after a change of variables.

3. Evaluating off-shell four-point Feynman integrals by Mellin–Barnes representation

Let us show how the above identity between the off shell triple box and tennis court can straightforwardly be obtained by means of the method of Mellin–Barnes (MB) representa-

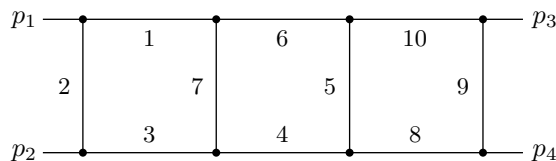


Figure 11: Labelled triple box.

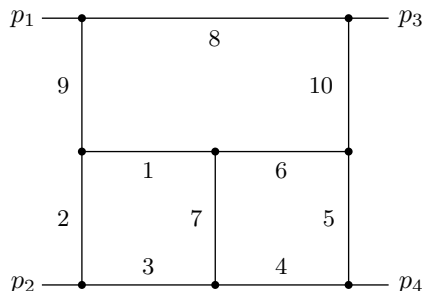


Figure 12: Labelled tennis court.

expressed in terms of gamma functions.) On the other hand, such a general derivation provides unambiguous prescriptions for choosing integration contours (see details in [27]). So, we consider the off shell triple box and tennis court labelled as shown in figures 11 and 12, with general powers of the propagators and one irreducible numerator in tennis court chosen as $[(l_1 + l_3)^2]^{-a_{11}}$, where $l_{1,3}$ are the momenta flowing through lines 1 and 3 in the same direction.

Experience shows that a minimal number of MB integrations for planar diagrams is achieved if one introduces MB integrations loop by loop, i.e. one derives a MB representation for a one-loop subintegral, inserts it into a higher two-loop integral, etc. This straightforward strategy provides the following 15-fold MB representations for the dimensionally regularised off-shell triple box and tennis court with general indices:

$$\begin{aligned}
 T_1(a_1, \dots, a_{10}; s, t, p_1^2, p_2^2, p_3^2, p_4^2; \epsilon) &= \frac{(i\pi^{d/2})^3 (-1)^a (-s)^{6-a-3\epsilon}}{\prod_{j=2,4,5,6,7,9} \Gamma(a_j) \Gamma(4-2\epsilon-a_{4,5,6,7})} \\
 &\times \frac{1}{(2\pi i)^{15}} \int_{-i\infty}^{+i\infty} \prod_{j=1}^{15} dz_j \frac{(-p_1^2)^{z_{12}} (-p_2^2)^{z_{13}} (-p_3^2)^{z_{4,9,14}} (-p_4^2)^{z_{5,10,15}} (-t)^{z_{11}}}{(-s)^{z_{4,5,9,10,11,12,13,14,15}}} \\
 &\times \frac{\Gamma(a_9 + z_{11,12,13}) \Gamma(a_7 + z_{1,2,3}) \Gamma(2 - \epsilon - a_{5,6,7} - z_{1,2,4}) \Gamma(2 - \epsilon - a_{4,5,7} - z_{1,3,5})}{\Gamma(a_1 - z_2) \Gamma(a_3 - z_3) \Gamma(4 - 2\epsilon - a_{1,2,3} + z_{1,2,3})} \\
 &\times \frac{\Gamma(a_5 + z_{1,4,5}) \Gamma(a_{4,5,6,7} + \epsilon - 2 + z_{1,2,3,4,5}) \Gamma(z_{11,14,15} - z_6)}{\Gamma(a_8 - z_7) \Gamma(a_{10} - z_8) \Gamma(4 - 2\epsilon - a_{8,9,10} + z_{6,7,8})} \\
 &\times \Gamma(2 - \epsilon - a_{8,9} + z_{6,7} - z_{11,12,14}) \Gamma(2 - a_{2,3} - \epsilon + z_{1,3} - z_{6,8,10}) \\
 &\times \Gamma(a_{8,9,10} + \epsilon - 2 + z_{11,12,13,14,15} - z_{6,7,8}) \Gamma(2 - \epsilon - a_{9,10} + z_{6,8} - z_{11,13,15}) \\
 &\times \Gamma(a_2 + z_{6,7,8}) \Gamma(2 - \epsilon - a_{1,2} + z_{1,2} - z_{6,7,9})
 \end{aligned}$$

$$\times \Gamma(z_{6,9,10} - z_1) \Gamma(a_{1,2,3} + \epsilon - 2 - z_{1,2,3} + z_{6,7,8,9,10}) \prod_{j=2,3,4,5,7,\dots,15} \Gamma(-z_j); \quad (3.2)$$

$$\begin{aligned} T_2(a_1, \dots, a_{11}; s, t, p_1^2, p_2^2, p_3^2, p_4^2; \epsilon) &= \frac{(i\pi^{d/2})^3 (-1)^a (-s)^{6-a-3\epsilon}}{\prod_{j=2,4,5,6,7,9} \Gamma(a_j) \Gamma(4-2\epsilon-a_{4,5,6,7})} \\ &\times \frac{1}{(2\pi i)^{15}} \int_{-i\infty}^{+i\infty} \prod_{j=1}^{15} dz_j \frac{(-p_1^2)^{z_{12}} (-p_2^2)^{z_{13}} (-p_3^2)^{z_{5,10,14}} (-p_4^2)^{z_{15}+z_8} (-t)^{z_{11}}}{(-s)^{z_{5,8,10,11,12,13,14,15}}} \prod_{j=2}^{15} \Gamma(-z_j) \\ &\times \frac{\Gamma(a_9 + z_{11,12,13}) \Gamma(a_7 + z_{1,2,3}) \Gamma(2 - a_{5,6,7} - \epsilon - z_{1,2,4}) \Gamma(2 - a_{4,5,7} - \epsilon - z_{1,3,5})}{\Gamma(a_1 - z_2) \Gamma(a_3 - z_3) \Gamma(4 - 2\epsilon - a_{1,2,3} + z_{1,2,3}) \Gamma(a_{10} - z_7)} \\ &\times \frac{\Gamma(a_5 + z_{1,4,5}) \Gamma(a_{4,5,6,7} + \epsilon - 2 + z_{1,2,3,4,5}) \Gamma(2 - a_{2,3} - \epsilon + z_{1,3} - z_{6,8,10})}{\Gamma(8 - 4\epsilon - a - z_{5,6,8,10}) \Gamma(a_8 - z_{4,9}) \Gamma(a_{1,2,3,4,5,6,7,11} + 2\epsilon - 4 + z_{4,5,6,7,8,9,10})} \\ &\times \Gamma(6 - a + a_{10} - 3\epsilon - z_{5,6,7,8,10,11,12,14}) \Gamma(a + 3\epsilon - 6 + z_{5,6,8,10,11,12,13,14,15}) \\ &\times \Gamma(a_2 + z_{6,7,8}) \Gamma(2 - \epsilon - a_{1,2} + z_{1,2} - z_{6,7,9}) \Gamma(6 - 3\epsilon - a + a_8 - z_{4,5,6,8,9,10,11,13,15}) \\ &\times \Gamma(z_{6,9,10} - z_1) \Gamma(a_{1,2,3} + \epsilon - 2 - z_{1,2,3} + z_{6,7,8,9,10}) \\ &\times \Gamma(a_{1,2,3,4,5,6,7,11} + 2\epsilon - 4 + z_{4,5,6,7,8,9,10,11,14,15}). \end{aligned} \quad (3.3)$$

Here $a_{4,5,6,7} = a_4 + a_5 + a_6 + a_7$, $a = \sum a_i$, $z_{11,12,13} = z_{11} + z_{12} + z_{13}$, etc. Moreover, in contrast to the rest of the paper, the letters s and t denote, in these equations as well in other equations of this section, the usual Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$.

These representation are written for the Feynman integrals in Minkowski space. (This is rather convenient, in particular this allows one to put some of the legs on-shell.) The corresponding Euclidean versions are obtained by the replacements $-s \rightarrow s$, $-t \rightarrow t$, $-p_1^2 \rightarrow p_1^2, \dots$ and by omitting the prefactors $(-1)^a$ and i^3 .

To calculate the triple box we need, i.e. $T_1^{(0)} = T_1(1, \dots, 1)$ at $d = 4$, we simply set all the indices a_i to one. We cannot immediately set $\epsilon = 0$ because there is $\Gamma(-2\epsilon)$ in the denominator. The value of the integral is, of course, non-zero, so that some poles in ϵ arise due to the integration. To resolve the structure of poles one can apply Czakon's code [28], which provides the following value of the integral in the limit $\epsilon \rightarrow 0$ after relabelling the variables by $z_{10} \rightarrow z_2, z_{14} \rightarrow z_3, z_{15} \rightarrow z_4, z_{11} \rightarrow z_5, z_{12} \rightarrow z_6$:

$$\begin{aligned} T_1^{(0)} &= \frac{(i\pi^2)^3}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \prod_{j=1}^6 dz_j \frac{(-p_1^2)^{z_6} (-p_2^2)^{-1-z_5,6} (-p_3^2)^{-1-z_5,6} (-p_4^2)^{z_6} (-t)^{z_5}}{(-s)^{2-z_5}} \\ &\times \frac{\Gamma(1 + z_{3,4}) \Gamma(1 + z_1 - z_{3,4,5}) \Gamma(z_{2,3,4,5} - z_1) \Gamma(z_4 - z_6)}{\Gamma(1 + z_4 - z_6) \Gamma(1 + z_{2,4} - z_6) \Gamma(2 + z_{1,6} - z_{2,4}) \Gamma(2 + z_{3,5,6})} \prod_j \Gamma(-z_j) \\ &\times \Gamma(z_{2,4} - z_6)^2 \Gamma(1 + z_{1,6} - z_{2,4})^2 \Gamma(1 + z_{5,6}) \Gamma(1 + z_{3,5,6}). \end{aligned} \quad (3.4)$$

To calculate the tennis court we need, i.e. $T_2^{(0)} = T_2(1, \dots, 1, -1)$ at $d = 4$, we proceed like in the previous case. Czakon's code provides the following integral (after relabelling $z_{10} \rightarrow z_2, z_{14} \rightarrow z_3, z_{15} \rightarrow z_4, z_{11} \rightarrow z_5, z_{12} \rightarrow z_6$):

$$T_2^{(0)} = \frac{(i\pi^2)^3}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \prod_{j=1}^6 dz_j \frac{(-p_1^2)^{z_6} (-p_2^2)^{-1-z_5-z_6} (-p_3^2)^{-1-z_5-z_6} (-p_4^2)^{z_6} (-t)^{z_5}}{(-s)^{1-z_5}}$$

$$\begin{aligned} & \times \frac{\Gamma(1+z_{3,4})\Gamma(1+z_1-z_{3,4,5})\Gamma(z_{2,3,4,5}-z_1)\Gamma(z_4-z_6)}{\Gamma(1+z_4-z_6)\Gamma(1+z_1-z_{2,3,5,6})\Gamma(2+z_{3,5,6})\Gamma(2+z_{2,3,5,6})} \prod_j \Gamma(-z_j) \\ & \times \Gamma(z_1-z_{2,3,5,6})^2 \Gamma(1+z_{5,6})\Gamma(1+z_{3,5,6})\Gamma(1+z_{2,3,5,6})^2. \end{aligned} \tag{3.5}$$

Now the simple change of variables $z_2 \rightarrow -z_2 + z_1 - z_3 - z_4 - z_5$ in (3.5) leads to an expression identical to (3.4) up to a factor of s and we obtain the identity $T_2^{(0)} = sT_1^{(0)}$, which corresponds to the identity $\Phi^{(3)} = \Psi^{(3)}$ of the previous section. (Observe that the factor s here appears because the general integrals (3.2) and (3.3) are defined without the appropriate prefactors present in the definitions of $\Phi^{(3)}$ and $\Psi^{(3)}$.)

Let us stress that one can also apply the technique of MB representation in a similar way in various situations where a given four-point off-shell Feynman integral cannot be reduced to ladder integrals.

Acknowledgments

The work of V.S. was supported by the Russian Foundation for Basic Research through grant 05-02-17645. E.S. thanks S. Ananth, B. Eden, Ya. Stanev, and V. Velizhanin for useful discussions and acknowledges the warm hospitality of the Albert Einstein Institute for Gravitational Physics in Potsdam where part of this work was done.

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